Design of Coded Caching Schemes With Linear Subpacketizations Based on Injective Arc Coloring of Regular Digraphs

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Abstract—Coded caching is an effective technique to decongest the amount of traffic in the backhaul link. In such a scheme, each file hosted in the server is divided into a number of packets to pursue a low broadcasting rate based on the designed placements at each user's cache. However, the implementation complexity of this scheme increases with the number of packets. It is important to design a scheme with a small subpacketization level and a relatively low transmission rate. Recently, placement delivery array (PDA) was proposed to address the subpacketization bottleneck of coded caching. This paper investigates the design of PDA from a new perspective, i.e., the injective arc coloring of regular digraphs. It is shown that the injective arc coloring of a regular digraph can yield a PDA with the same number of rows and columns. Based on this, a new class of regular digraphs are defined and the upper bounds on the injective chromatic index of such digraphs are derived. Consequently, four new coded caching schemes with a linear subpacketization level and a relatively small transmission rate are proposed, one of which generalizes the existing scheme for the scenario with a more flexible number of users.

Index Terms—Coded caching, placement delivery array, regular digraph, injective arc coloring, subpacketization.

I. INTRODUCTION

THE dramatic increase of video streaming requests can easily cause severe network congestions during the peaktraffic times. One possible solution is to exploit the off-peak

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network resources, such as to cache some of the possibly demanded contents in users' local memories, i.e., the so called caches. This can help decrease the network traffic when the cached contents are requested. The gain offered by this approach is called *local gain*, which depends on the size of local caches. A more effective way of caching is through coding, which was first proposed by Maddah-Ali and Niesen [1]. It reduces the network pressure during the peak times by strategically designing the contents cached into the network users and the broadcast messages to obtain the global gain. In the centralized coded caching system, a central server containing N files of the same size is connected to K users over a noiseless shared link. Each user has a cache memory with a size of M files, where M < N. It operates in two phases: the placement phase during the off-peak times and the delivery phase during the peak times. In the placement phase, each file is divided into F equal packets, and each user's cache is filled with some form of these packets without any prior knowledge of future demands. The quantity F is referred to as the subpacketization level. In the delivery phase, each user reveals its requested file to the server. After receiving the user demands, the server transmits some coded symbols over a noiseless shared link to all the users so that their demands can be satisfied with the assistance of the locally cached contents. Normalizing the minimal worst case transmission load by the size of file would result in the so called *transmission rate* R, i.e., the minimum number of files that must be communicated so that any possible demands can be satisfied. Under such paradigm, if the packets are cached directly without coding in the placement phase, it is called an uncoded placement; otherwise, it is called a *coded placement*. We summarize the prior work as follows.

A. Prior Work

The original coded caching scheme proposed by Maddah-Ali and Niesen [1] is realized by a combinatorial design in the placement phase and a linear network coding in the delivery phase, which is referred to as the MN scheme. It achieves an optimal transmission rate under the constraints of uncoded placement and $K \leq N$ [2]. Observing that there exist some redundant transmissions in the MN scheme when a file is requested by several users, Yu et al. [3] proposed a scheme that improves upon the MN scheme and achieves an optimal

0090-6778 © 2023 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See https://www.ieee.org/publications/rights/index.html for more information. transmission rate under the constraint of uncoded placement. They further showed that the multiplicative gap between the optimal caching scheme with uncoded placement and any caching scheme with coded placement is at most two [4]. The MN scheme has been extensively studied over other network scenarios, such as the decentralized caching [5], the multi-level popularity and access [6], the combination networks [7], the device-to-device (D2D) caching systems [8] and the arbitrary multiserver linear networks [9].

There exist some work on reducing the subpacketization level of the MN scheme, but they usually trade it with the transmission rate [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. To the best of our knowledge, most existing schemes that aim to reduce the subpacketization level are constructed under uncoded placement. With a large number of users, the tradeoff between the subpacketization level and the transmission rate was first investigated by Shanmugam et al. in [17]. However, for an arbitrary number of users, it remains challenging to characterize such tradeoff. It was shown in [18] that for the fixed transmission rate, the MN scheme has a minimum subpacketization level. It can be seen that its subpacketization level increases exponentially with the number of users, which makes it impractical for large networks. Therefore, it is important to reduce the subpacketization level of the MN scheme, while maintaining a relatively low transmission rate. In particular, Yan et al. [15] represented the coded caching scheme by an array called the placement delivery array (PDA) that is proved to be effective for reducing the subpacketization level. It has been shown that the MN scheme can be considered as a special class of PDAs. Through constructing PDAs, two classes of coded caching schemes were proposed with a reduced subpacketization level over that of the MN scheme. But they yield a slightly increased transmission rate. Since then, PDA has been utilized as a systematic approach to design the coded caching schemes that yield a low subpacketization level [16], [18], [19], [20], [21], [22], [23], [24], [25], [26]. Recently, the multiple antennas coded caching scenario was also considered to reduce the subpacketization level [28], [29], [30]. It is another important research direction in this area. Motivated by the PDA based coded caching, Yang et al. [31] proposed a transformation approach to construct a multiple antennas coded caching scheme from a special class of PDAs. A multiple antennas coded caching scheme with a low subpacketization level can be designed through constructing proper PDAs. Other combinatorial methods for reducing the subpacketization include Ruzsa-Szemerédi graphs [14], projective geometry and line graphs [11], hypergraphs [13] and combinatorial design [10]. Table I summarizes the existing schemes with the advantages in either the subpacketization level or the transmission rate.

One of the important questions in coded caching design is how to realize a linear subpacketization level so that the subpacketization increases linearly with the number of users. It has been shown that with a small memory ratio requirement, a coded caching scheme with a linear subpacketization level can be constructed at a near constant transmission rate [14]. However, it requires an extremely large number of users. The MN scheme achieves F = K when $\frac{M}{N} = \frac{1}{K}$, but the transmission rate would be $R = \frac{K-1}{2}$. The schemes of [24] can yield a linear subpacketization level and a small transmission rate. However, they require the number of users to be some non-flexible values. A more recent scheme of [11] achieves a linear subpacketization level but with a larger transmission rate and a non-flexible user number. Therefore, most coded caching schemes that yield a linear subpacketization level are disadvantageous in either the number of users or the transmission rate.

B. Contribution and Organization of This Work

This paper considers the PDA construction from the perspective of graph coloring, aiming to design a coded caching scheme that can work for a flexible number of users, meanwhile achieve a linear subpacketization level and a relatively small transmission rate. Our key technical contributions include:

• We propose the PDA design for coded caching through the injective arc coloring of regular digraphs. It is shown that the injective arc coloring of a regular digraph can yield a PDA with the same number of rows and columns. This enables the design of PDA to utilize the existing structures of regular digraphs. By observing that the strong edge coloring of regular graphs can be viewed as a special injective arc coloring of regular digraphs, a new coded caching scheme with a linear subpacketization level can be obtained from the existing strong edge coloring of unitary Cayley graphs. This scheme will be characterized in *Theorem 4*.

• We also define a new class of regular digraphs and derive upper bounds on the injective chromatic index of such digraphs. Consequently, a new coded caching scheme that can accommodate a flexible number of users is obtained. It yields a linear subpacketization level and a relatively small transmission rate. This scheme will be characterized in *Theorem 2*. Based on the proposed coded caching scheme, this research finds out that some packets cached by the users have no multicast opportunities in the delivery phase. By utilizing the maximum distance separable (MDS) code in the placement phase, a new coded caching scheme with a smaller subpacketization level and memory ratio is further proposed. It generalizes the scheme of [24] and supports a more flexible number of users. This scheme will be characterized in *Theorem 5*.

The rest of this paper is organized as follows. In Section II, we briefly review the background of the centralized coded caching system. The relationship between PDA and injective arc coloring of regular digraphs is presented in Section III. Section IV proposes four new PDA schemes that are designed through the injective arc coloring of regular digraphs. Their performance analyses are given in Section V. Finally, Section VI concludes the paper.

II. SYSTEM MODEL AND THE PDA

This section presents the coded caching system model and the PDA. Some key notations are introduced as follows.

Schemes and Parameters	User Number K	Caching Ratio $\frac{M}{N}$	Rate R	Subpacketization Level F
MN scheme in [1], any $k, t \in \mathbb{N}^+$	k	$\frac{t}{L}$	$\frac{k-t}{1+t}$	$\binom{k}{t}$
with $t < k$		ĸ	1+1	(1)
MN scheme with grouping in [17],	. 1.	t	c(k-t)	(k)
any $k, t, c \in \mathbb{N}^+$ with $t < k$	СК	$\overline{\overline{k}}$	1+t	(t)
Scheme in [16], any $a, b, m, \lambda \in \mathbb{N}^+$	(m)	$\begin{pmatrix} a \\ \lambda \end{pmatrix} \begin{pmatrix} m-a \\ b-\lambda \end{pmatrix}$	$\binom{m}{a+b-2\lambda}\binom{a+b-2\lambda}{a-\lambda}$	(m)
with $a < m, b < m$ and $\lambda < \min\{a, b\}$	(_a)	$1 - \frac{m}{\binom{m}{b}}$	$\left(\begin{array}{c} (m) \\ (m) \\ (b) \end{array} \right)$	$\left(\begin{smallmatrix} b \end{smallmatrix} \right)$
Scheme in [11], any $n, m, k \in \mathbb{N}^+$,	$\left q^{\frac{n(n-1)}{2}} \prod_{i=0}^{n-1} \begin{bmatrix} k-i \\ 1 \end{bmatrix}_{q} \right $	$1 - q^{mn} \cdot $	$\frac{m!q^{mn}}{(m+n)!}q^{\frac{n(n-1)}{2}}.$	$q^{\frac{m(m-1)}{2}} \prod_{i=0}^{m-1} \begin{bmatrix} k-i \\ 1 \end{bmatrix}_{q}$
prime power q with $n+m \leq k$	<u></u>	$\prod_{i=0}^{n-1} \frac{\lfloor 1 \rfloor_q}{\lfloor k-i \rfloor_q}$	$\left \prod_{i=0}^{n-1} \left\lfloor \substack{k-m-i\\1} \right\rfloor_q\right $	
Scheme in [24], any $n, w' \in \mathbb{N}^+$	n^n	$\begin{pmatrix} n \\ w' \end{pmatrix}$	$\binom{n}{w'}2^{n-w'}$	$\sum w'(n)$
with $w' < n$	2.0	$1 - \frac{\overline{\sum_{i=0}^{w'} \binom{n}{i}}}{\sum_{i=0}^{w'} \binom{n}{i}}$	$\sum_{i=0}^{w'} \binom{n}{i}$	$\sum_{i=0} (i)$
Scheme in [14], $\epsilon(\delta) \to 0$ and	$k(\delta)$	$L - \epsilon(\delta)$	L^{δ}	$k(\delta)$
$k(\delta) \to \infty \text{ as } \delta \to 0$		n v	n	(0)
Scheme in [23], any $r, k, z \in \mathbb{N}^+$	$2^r k$	$1 - \frac{r+1}{2^r} + \frac{rz}{2^rk}$	$\frac{k^2 + k^2 r - krz}{2^r k}$	2^rk

 TABLE I

 Summary of the Existing Coded Caching Schemes

* Note: $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\alpha}-1)\cdots(q^{\alpha-\beta+1}-1)}{(q^{\beta}-1)\cdots(q-1)}$ for any positive integers α, β and prime power q.

Notations: Let calligraphic symbols, bolded capital letters and bolded lower-case letters denote sets, arrays and vectors, respectively. Symbol \oplus represents the exclusive-or (XOR) operation. Let \mathbb{Z}_p denote the ring of integers modulo p. Let \mathbb{Z}_p^n further denote a set of vectors whose elements are obtained by n-fold Cartesian product of \mathbb{Z}_p , i.e., $\mathbb{Z}_p^n = \{\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \mid (x_0, x_1, \dots, x_{n-1}) \in \mathbb{Z}_p \times \mathbb{Z}_p \times \dots \times \mathbb{Z}_p\}$. We use $|\cdot|$ to denote the cardinality of

a set. Let \mathbb{N}^+ denote the set of positive integers. The set of consecutive integers is denoted as $[x:y] = \{x, x+1, \ldots, y\}$. Given an array **P**, let $\mathbf{P}[i, j]$ denote its entry of row *i* and column *j*. Let $\binom{[0:m-1]}{t}$ denote the collection of all subsets of [0:m-1] with size *t*, i.e., $\binom{[0:m-1]}{t} = \{S \mid S \subseteq [0:m-1], |S| = t\}$. For a length-*m* vector **x** and a set $S \subseteq [0:m-1]$, let $\mathbf{x}|_S$ denote a vector obtained by taking the coordinate indexed by $j \in S$. Finally, the vectors in examples are written as strings, e.g., (1, 1, 1, 1) is written as 1111.

A. Centralized Coded Caching System

In a centralized coded caching system, a server containing N files of the same size is connected to K users through a noiseless shared link, as shown in Fig. 1. Each user is equipped with a dedicated cache with a size of M files, where M < N. The N files and K users are denoted by $\mathcal{W} = \{W_0, W_1, \ldots, W_{N-1}\}$ and $\mathcal{K} = [0: K-1]$, respectively. An F-division (K, M, N) coded caching scheme consists of two phases, which are described as follows.

• Placement Phase: Each file is divided into F equal packets i.e., $W_n = \{W_{n,j} | j \in [0:F-1]\}, n \in [0:N-1]$. The server places some packets (or coded packets) directly into each user's cache without any prior knowledge of the demands in the delivery phase. Let Z_k denote the contents cached by user k, where $k \in \mathcal{K}$. The size of Z_k cannot be greater than the capacity of each user's cache memory size M.



Fig. 1. Coded caching system.

• Delivery Phase: Each user requests an arbitrary file from \mathcal{W} . The request vector is denoted by $\mathbf{d} = (d_0, d_1, \dots, d_{K-1})$, i.e., user k requests file W_{d_k} , where $k \in \mathcal{K}$ and $d_k \in [0 : N - 1]$. Once the server receives the request vector \mathbf{d} , it broadcasts a signal of at most RF packets such that all the users can correctly decode their requested files together with the cached contents.

B. Placement Delivery Array

Let us review the definition of PDA that can be used to characterize both the placement phase and the delivery phase.

Definition 1 [15]: Given $K, F, Z, S \in \mathbb{N}^+$, an $F \times K$ array $\mathbf{P} = (\mathbf{P}[i, j])$, where $i \in [0 : F - 1]$, $j \in [0 : K - 1]$, and $\mathbf{P}[i, j] \in [0 : S - 1] \cup \{*\}$, is called a (K, F, Z, S) PDA if it satisfies the following conditions:

C1. Symbol "*" appears exactly Z times in each column;

C2. Each integer of [0: S-1] appears at least once in the array;

C3. For any two distinct entries $\mathbf{P}[i_1, j_1]$ and $\mathbf{P}[i_2, j_2]$, $\mathbf{P}[i_1, j_1] = \mathbf{P}[i_2, j_2] = s$ is an integer only if

(a). $i_1 \neq i_2, j_1 \neq j_2$, i.e., they lie in distinct rows and distinct columns;

$$\begin{pmatrix} s & * \\ * & s \end{pmatrix}, \quad \begin{pmatrix} * & s \\ s & * \end{pmatrix}.$$

Algorithm 1 was proposed to realize the PDA based coded caching schemes. Given a (K, F, Z, S) PDA **P** with column indices representing the users and row indices representing the packets, $\mathbf{P}[j,k] = *$ implies the server has placed the *j*th packet of all the files into the cache of user k. Condition C1 of *Definition* 1 implies that each user has the same memory size and the memory ratio is $\frac{M}{N} = \frac{Z}{F}$. If $\mathbf{P}[j,k] = s$, where $s \in [0: S-1]$, it indicates that user k does not cache the jth packet of all the files. The linear combination of the requested packets indicated by s will be broadcast by the server at time slot s. Condition C3 of *Definition* 1 ensures the decodability, since it has cached all the other packets in the multicast message except its requested one. Finally, Condition C2 of Definition 1 implies that the number of messages transmitted by the server is exactly S and the transmission rate is $R = \frac{S}{E}$. Furthermore, the coding gain in each time slot $s \in [0: S-1]$, denoted by g_s , is the occurrence number of integer s in **P**, since the coded packet broadcast at time slot s is beneficial to g_s users. Based on Algorithm 1, an F-division (K, M, N)coded caching scheme can be characterized by Lemma 1.

Algorithm 1 Coded Caching Scheme Based on PDA [15]1: Procedure Placement (P, W)2: Split each file $W_n \in W$ into F packets as $W_n = \{W_{n,j} \mid j \in [0:F-1]\}$.3: For $k \in \mathcal{K}$ do4: $\mathcal{Z}_k \leftarrow \{W_{n,j} \mid \mathbf{P}[j,k] = *, \forall n \in [0:N-1]\}$;5: Procedure Delivery (P, W, d)6: For $s = 0, 1, \dots, S - 1$ do7: Server sends $\oplus_{\mathbf{P}[j,k]=s,j \in [0:F-1],k \in [0:K-1]} W_{d_k,j}$.

Lemma 1 [15]: Given a (K, F, Z, S) PDA, there always exists an F-division (K, M, N) coded caching scheme with a memory ratio of $\frac{M}{N} = \frac{Z}{F}$ and a transmission rate of $R = \frac{S}{F}$.

The following example demonstrates this property.

Example 1: Given a (4, 4, 2, 4) PDA **P**, and based on Algorithm 1, a 4-division (4, 2, 4) coded caching scheme can be obtained as

$$\mathbf{P} = \begin{pmatrix} 0 & * & * & 3\\ * & 1 & 2 & *\\ 1 & * & * & 2\\ * & 0 & 3 & * \end{pmatrix}.$$
 (1)

• Placement Phase: Each file W_n is divided into four packets, i.e., $W_n = \{W_{n,0}, W_{n,1}, W_{n,2}, W_{n,3}\}$, where $n \in [0:3]$. The contents cached by each user are

$$\begin{aligned} \mathcal{Z}_0 &= \{ W_{n,1}, W_{n,3} \mid n \in [0:3] \}, \\ \mathcal{Z}_1 &= \{ W_{n,0}, W_{n,2} \mid n \in [0:3] \}, \\ \mathcal{Z}_2 &= \{ W_{n,0}, W_{n,2} \mid n \in [0:3] \}, \\ \mathcal{Z}_3 &= \{ W_{n,1}, W_{n,3} \mid n \in [0:3] \}. \end{aligned}$$

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• Delivery Phase: Let us assume that the request vector is $\mathbf{d} = (0, 1, 2, 3)$. The signals sent by the server at the four time slots (TSs) are listed as follows. TS-0: $W_{0,0} \oplus W_{1,3}$; TS-1: $W_{0,2} \oplus W_{1,1}$; TS-2: $W_{2,1} \oplus W_{3,2}$; TS-3: $W_{2,3} \oplus W_{3,0}$. Each user can then reconstruct its required file. E.g., user 0 requires W_0 and it has cached $W_{0,1}$ and $W_{0,3}$. At TS-0, it can obtain $W_{0,0}$ with its received coded packet $W_{0,0} \oplus W_{1,3}$, where $W_{1,3}$ was cached. At TS-1, it can obtain $W_{0,2}$ with its received coded packet $W_{0,2} \oplus W_{1,1}$, where $W_{1,1}$ was cached. Hence, the transmission rate is $R = \frac{4}{4} = 1$.

III. INJECTIVE ARC COLORING OF REGULAR DIGRAPHS AND ITS RELATION TO PDA

This section investigates the design of a PDA with the same number of rows and columns through the injective arc coloring of a regular digraph. Some graph theoretic notations are reviewed as follows. Let $G = G(\mathcal{V}, \mathcal{E})$ denote a simple undirected graph with vertex set \mathcal{V} and edge set \mathcal{E} . The degree of a vertex v in a graph G is denoted by d(v). A graph G is called r-regular if d(v) = r for all $v \in \mathcal{V}$. Let $D = D(\mathcal{V}, \mathcal{E})$ denote a digraph with vertex set \mathcal{V} and arc set \mathcal{E} . For a vertex $v \in \mathcal{V}$, we denote the indegree and outdegree of v by $d^{-}(v)$ and $d^{+}(v)$, respectively. If $d^{-}(v) = d^{+}(v) = r$ for each vertex $v \in \mathcal{V}$, D is called a r-regular digraph. In this paper, we focus on the regular digraph with reverse arcs but without a directed self loop. For clarity, we introduce several definitions of graph coloring.

Definition 2 [32]: For a graph G, a proper edge coloring is an assignment of colors to each edge of a graph such that no two edges with a common endpoint receive the same color. The smallest number of colors needed in a proper edge coloring of a graph G is called the chromatic index of G.

Definition 3 [32]: A strong edge coloring is a proper edge coloring, with the further condition that no two edges with the same color lie on a path of length three. The strong chromatic number is the minimum number of colors that allow a strong edge coloring, denoted by $\chi_s(G)$.

The concept of injective edge coloring was first introduced by Cardoso et al. [33]. Its definition is described as follows.

Definition 4 [33]: An edge coloring of a graph G is injective if any two edges e and f that are at a distance of exactly one (i.e., there exists an edge between two edges e and f) or in a common triangle receive distinct colors. The injective chromatic index of G, denoted by $\chi_i(G)$, is the minimum number of colors needed for an injective edge coloring of G.

Based on *Definitions 3* and 4, it can be seen that the condition of strong edge coloring of a graph G imposes more constraints than that of injective edge coloring. E.g., given a path $P = v_0v_1v_2v_3$ with vertex set $\mathcal{V} = \{v_0, v_1, v_2, v_3\}$, if edges v_0v_1 , v_1v_2 and v_2v_3 are assigned with colors 0, 1 and 2, respectively, this coloring is a strong edge coloring of P. It can be seen that this coloring is also an injective edge coloring. However, it is not a strong edge coloring. Therefore, a strong edge coloring of a graph G is an injective edge coloring, but not vice versa. In this paper, we extend the



Fig. 2. 3-regular digraph with the number of injective arc coloring of 6.

definition of injective edge coloring of graph G to the regular digraph D as follows.

Definition 5: An arc coloring of a regular digraph D is injective if any two arcs **e** and **f** that are at a distance of exactly one (i.e., there exists an arc between two arcs **e** and **f** in a directed path) or in a common directed triangle receive distinct colors. The injective chromatic index of D, denoted by $\chi_i(D)$, is the minimum number of colors needed for an injective arc coloring of D.

The following example illustrates the above definition.

Example 2: Given the arc-colored regular digraph of Fig. 2, and based on *Definition 5*, it can be observed that this coloring is injective since any two arcs have distinct colors if they are at a distance of exactly one or in a common directed triangle.

Based on the above introduction, given an injective arccolored regular digraph D with vertex set \mathcal{V} and arc set \mathcal{E} , if the arcs in \mathcal{E} are colored by colors $0, 1, \ldots, S-1$, we can construct a $|\mathcal{V}| \times |\mathcal{V}|$ array $\mathbf{P} = (\mathbf{P}[v_j, v_k]) \ (v_j, v_k \in \mathcal{V})$ containing alphabet set $[0: S-1] \cup \{*\}$ as

$$\mathbf{P}[v_j, v_k] = \begin{cases} *, & \text{if } (v_j, v_k) \notin \mathcal{E}; \\ s, & \text{if } (v_j, v_k) \in \mathcal{E} \text{ and it is colored by } s. \end{cases}$$
(2)

The following *Example* 3 is further developed based on *Example* 2, in order to illustrate the above observation.

Example 3: Let D denote an arc-colored regular digraph with vertex set \mathcal{V} and arc set \mathcal{E} , as shown in Fig. 2. Based on (2), for any $v_j, v_k \in \mathcal{V}$, we have $\mathbf{P}[v_j, v_k] = *$, if $(v_j, v_k) \notin \mathcal{E}$; and $\mathbf{P}[v_j, v_k] = s$, if $(v_j, v_k) \in \mathcal{E}$ and it is colored by s. E.g., we have $\mathbf{P}[v_0, v_0] = *$ since D has no directed self loop; and $\mathbf{P}[v_0, v_1] = 1$ since the arc (v_0, v_1) is assigned with color 1. As a result, the following array \mathbf{P} can be obtained. It can be seen that \mathbf{P} is a (8, 8, 5, 6) PDA.

		v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
	v_0	(*	1	3	5	*	*	*	*)
	v_1	0	*	*	*	3	5	*	*
	v_2	2	*	*	*	1	*	5	*
р_	v_3	4	*	*	*	*	1	3	*
г =	v_4	*	2	0	*	*	*	*	5
	v_5	*	4	*	0	*	*	*	3
	v_6	*	*	4	2	*	*	*	1
	v_7	* /	*	*	*	4	2	0	* /

Based on the above investigation, the following theorem that describes the relationship between the injective arc coloring of a regular digraph and a PDA that has the same number of rows and columns can be reached.

Theorem 1: For any injective arc-colored regular digraph D with K vertices and K - Z indegrees, if the arcs of D can be colored by the colors $0, 1, \ldots, S - 1$, the corresponding array **P** is a (K, K, Z, S) PDA.

Proof: Given an injective arc coloring of regular digraph with K vertices, one needs to show that the resulting array satisfies the definition of PDA. Since D is a regular digraph, each vertex has the same indegree and outdegree. Let us assume that the indegree of each vertex is K - Z. Based on (2), each column of **P** has Z "*"s. Furthermore, it is impossible for an entry to appear more than once in each row or each column. This is because any two arcs with the same head or tail will receive distinct colors due to its injective arc coloring. If there exist two distinct entries such that $\mathbf{P}[v_{j_1}, v_{k_1}] = \mathbf{P}[v_{j_2}, v_{k_2}] = s$, arcs (v_{j_1}, v_{k_1}) and (v_{j_2}, v_{k_2}) are colored by s. Note that this coloring is an injective arc coloring and D is a digraph without a directed self loop. This implies that (v_{k_1}, v_{j_2}) and (v_{k_2}, v_{j_1}) are not arcs of D. Since the regular digraph has reverse arcs, it can be seen that both (v_{j_1}, v_{k_2}) and (v_{j_2}, v_{k_1}) are not the arcs of D, i.e., $\mathbf{P}[v_{j_1}, v_{k_2}] = \mathbf{P}[v_{j_2}, v_{k_1}] = *$. Hence, the array \mathbf{P} defined in (2) is a (K, K, Z, S) PDA.

This provides a new method to construct PDAs from the arc injective coloring of regular digraphs. The PDA characterized by *Theorem 1* can realize a coded caching scheme with a transmission rate of $R = \frac{S}{K}$. Given K and Z, it is desirable to obtain a PDA scheme that can yield a transmission rate as small as possible. This implies that the number of colors S needed for an injective arc coloring should be as small as possible. Therefore, it is important to properly construct a regular digraph and determine its optimal or near optimal number of injective arc coloring, which will be discussed in the following Section IV.

IV. THE NEW PDA SCHEMES

This section proposes three new PDA constructions via injective arc coloring of regular digraphs. The designs can realize a coded caching scheme that supports a flexible number of users with a linear subpacketization level and a relatively small transmission rate.

A. A New Construction of Regular Digraphs and Their Injective Arc Colorings

This subsection defines a new class of regular digraphs and proposes a novel arc coloring rule for them such that their resulting colorings are injective. Given two vectors \mathbf{x} and \mathbf{y} , the Hamming distance between \mathbf{x} and \mathbf{y} is defined as the number of coordinates that \mathbf{x} and \mathbf{y} differ, and denoted as $d_{\mathrm{H}}(\mathbf{x}, \mathbf{y})$. Consequently, a regular digraph \hat{D} with the number of vertices $p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}$ can be defined as follows.

Definition 6: Given any $w, n_i, m \in \mathbb{N}^+$ and distinct positive integers $p_0, p_1, \ldots, p_{m-1}$ with $p_i \ge 2$ and $w < n_0 + n_1 + \cdots + n_{m-1}$ for $i \in [0 : m-1]$, the vertex set \mathcal{V} and arc set \mathcal{E} of D are defined as

$$\mathcal{V} = \{ \mathbf{x} = (\underbrace{x_0^{(p_0)}, x_1^{(p_0)}, \dots, x_{n_0-1}^{(p_0)}}_{n_0}, \underbrace{x_0^{(p_1)}, x_1^{(p_1)}, \dots, x_{n_{1-1}}^{(p_{1-1})}, \dots, \underbrace{x_0^{(p_{m-1})}, x_1^{(p_{m-1})}, \dots, x_{n_m-1}^{(p_{m-1})}}_{n_{m-1}} \\ \times \underbrace{\mathbf{x} \in \mathbb{Z}_{p_0}^{n_0} \times \mathbb{Z}_{p_1}^{n_1} \times \dots \times \mathbb{Z}_{p_{m-1}}^{n_{m-1}} \},$$

and

$$\mathcal{E} = \{ (\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{x}) \mid d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = w, \mathbf{x}, \mathbf{y} \in \mathcal{V} \},\$$

respectively.

In the following, we will present a novel injective arc coloring rule for \hat{D} . To do so, its arcs are partitioned into several disjoint subsets. A color is assigned to each of the arc partition such that the resulting coloring is an injective arc coloring. Given any arc $(\mathbf{x}, \mathbf{y}) \in \mathcal{E}$, let $\mathcal{C}_{\mathbf{x}-\mathbf{y}}$ denote the set of the coordinates where two vectors \mathbf{x} and \mathbf{y} differ, i.e., $\mathcal{C}_{\mathbf{x}-\mathbf{y}} = \{(j^{(i)}, p_i) \mid x_{j^{(i)}}^{(p_i)} \neq y_{j^{(i)}}^{(p_i)}, j^{(i)} \in [0 : n_i - 1], i \in [0 : m - 1]\}$. Let $\mathcal{C}_{\mathbf{e}}$ denote the set of nonzero coordinates in \mathbf{e} , where $\mathbf{e} = \mathbf{x} - \mathbf{y}$ and $|\mathcal{C}_{\mathbf{e}}| = w$. Let

$$\mathcal{T}_{\mathbf{e}} = \{ \mathbf{t} \mid \mathbf{t} \in \mathbb{Z}_{p_{i_0}} \times \mathbb{Z}_{p_{i_1}} \times \cdots \times \mathbb{Z}_{p_{i_{w-1}}}, i_0 \le i_1 \le \cdots \le i_{w-1} \}$$

denote a set of vectors with length w, where nonnegative integers $i_0, i_1, \ldots, i_{w-1}$ are determined by $C_{\mathbf{e}} = \{(j_{\alpha_0}^{(i_0)}, p_{i_0}), (j_{\alpha_1}^{(i_1)}, p_{i_1}), \ldots, (j_{\alpha_{w-1}}^{(i_{w-1})}, p_{i_{w-1}})\}$ and $j_{\alpha_u}^{(i_u)} \in [0:$ $n_{i_u} - 1]$ for $u \in [0: w - 1]$. The arc set \mathcal{E} of \hat{D} can be partitioned as

$$\mathcal{E} = \bigcup_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} \mathcal{E}_{\mathbf{e}} = \bigcup_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} \bigcup_{\mathbf{t} \in \mathcal{T}_{\mathbf{e}}} \mathcal{E}_{\mathbf{e}, \mathbf{t}}, \quad (3)$$

where

$$\begin{split} \mathcal{E}_{\mathbf{e}} &= \{(\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}, \mathbf{x} - \mathbf{y} = \mathbf{e}\},\\ \mathcal{E}_{\mathbf{e}, \mathbf{t}} &= \{(\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}, \mathbf{x} - \mathbf{y} = \mathbf{e}, \mathbf{y}|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{t}\} \end{split}$$

and the computations in the *i*th block are performed under modulo p_i with $i \in [0: m-1]$.

The above arc partition leads to the following result.

Proposition 1: The assignment of a distinct color for each subset $\mathcal{E}_{e,t}$ forms an injective arc coloring of \hat{D} .

Proof: Let us consider $\mathcal{E}_{e,t}$ for any e and t defined above. It can be seen that $|\mathcal{E}_{e,t}| > 1$. Based on (3), for any two distinct arcs $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \mathcal{E}_{e,t}$, we have

$$\begin{cases} \mathbf{x}_1 - \mathbf{y}_1 = \mathbf{x}_2 - \mathbf{y}_2 = \mathbf{e}, \\ \mathbf{y}_1|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{y}_2|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{t}. \end{cases}$$

This implies that $\mathbf{x}_1|_{\mathcal{C}_e} = \mathbf{x}_2|_{\mathcal{C}_e}$. Hence, $d_H(\mathbf{x}_1, \mathbf{y}_2) \geq d_H(\mathbf{x}_1|_{\mathcal{C}_e}, \mathbf{y}_2|_{\mathcal{C}_e}) = d_H(\mathbf{x}_2|_{\mathcal{C}_e}, \mathbf{y}_2|_{\mathcal{C}_e}) = w$. Let \mathcal{C}'_e denote the set of zero coordinates in **e**. If $d_H(\mathbf{x}_1, \mathbf{y}_2) = w$, we have $\mathbf{x}_1|_{\mathcal{C}'_e} = \mathbf{y}_2|_{\mathcal{C}'_e}$ since $d_H(\mathbf{x}_1, \mathbf{y}_2) = d_H(\mathbf{x}_1|_{\mathcal{C}_e}, \mathbf{y}_2|_{\mathcal{C}_e}) + d_H(\mathbf{x}_1|_{\mathcal{C}_e'}, \mathbf{y}_2|_{\mathcal{C}_{e'}}) = d_H(\mathbf{x}_1|_{\mathcal{C}_e}, \mathbf{y}_1|_{\mathcal{C}_e}) + d_H(\mathbf{x}_1|_{\mathcal{C}_{e'}}, \mathbf{y}_2|_{\mathcal{C}_{e'}}) = w$. Hence, $\mathbf{x}_1|_{\mathcal{C}'_e} = \mathbf{x}_2|_{\mathcal{C}'_e} = \mathbf{y}_1|_{\mathcal{C}'_e} = \mathbf{y}_2|_{\mathcal{C}'_e}$ and we have $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{y}_1 = \mathbf{y}_2$, which contradicts the

hypothesis. Therefore, $d_{\rm H}(\mathbf{x}_1, \mathbf{y}_2) > w$. Similarly, we also have $d_{\rm H}(\mathbf{x}_2, \mathbf{y}_1) > w$. This implies that $(\mathbf{x}_1, \mathbf{y}_2)$, $(\mathbf{y}_2, \mathbf{x}_1)$, $(\mathbf{x}_2, \mathbf{y}_1)$ and $(\mathbf{y}_1, \mathbf{x}_2)$ are not arcs of \hat{D} . Therefore, the assignment of a distinct color for each subset $\mathcal{E}_{\mathbf{e}, \mathbf{t}}$ forms an injective arc coloring.

The following example illustrates the above property.

Example 4: Given $m = 2, w = 1, p_0 = 2, p_1 = 3, n_0 = 2$ and $n_1 = 1$, based on *Definition 6*, a regular digraph can be constructed by defining its vertex set \mathcal{V} and arc set \mathcal{E} as

$$\mathcal{V} = \{ \mathbf{x} = (\underbrace{x_0^{(2)}, x_1^{(2)}}_{2}, \underbrace{x_0^{(3)}}_{1}) \mid (x_0^{(2)}, x_1^{(2)}, x_0^{(3)}) \in \mathbb{Z}_2^2 \times \mathbb{Z}_3 \}$$

= {000, 001, 002, 100, 101, 102, 010, 011, 012,
110, 111, 112}

and $\mathcal{E} = \{(\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{x}) \mid d_{H}(\mathbf{x}, \mathbf{y}) = 1, \mathbf{x}, \mathbf{y} \in \mathcal{V}\} = \{(000, 002), (002, 000), (002, 001), (001, 002), (001, 101), (101, 001), (102, 101), (101, 102), (102, 100), (100, 102), (100, 000), (000, 100), (000, 001), (001, 000), (100, 101), (101, 100), (002, 102), (102, 002), (002, 012), (012, 002), (001, 011), (011, 001), (000, 010), (010, 000), (112, 102), (102, 112), (110, 112), (112, 110), (112, 012), (012, 112), (012, 010), (010, 012), (010, 011), (011, 011), (011, 011), (111, 110), (110, 111), (111, 111), (1$

$$\mathcal{E}_{010} = \{ (010,000), (011,001), (012,002), (110,100), (111, 101), (112,102), (000,010), (001,011), (002,012), (100,110), (101,111), (102,112) \}.$$

Note that for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{E}_{010}$, we have $\mathcal{C}_{\mathbf{x}-\mathbf{y}} = \mathcal{C}_{010} = \{(j^{(i)}, p_i) \mid x_{j^{(i)}}^{(p_i)} \neq y_{j^{(i)}}^{(p_i)}, j^{(i)} \in [0 : n_i - 1], i \in [0 : 1]\} = \{(j_{\alpha_0}^{(0)}, p_0)\} = \{(1, 2)\}$. This implies that $\mathcal{T}_{010} = \{\mathbf{t} \mid \mathbf{t} \in \mathbb{Z}_{p_0}\} = \{\mathbf{t} \mid \mathbf{t} \in \mathbb{Z}_2\} = \{0, 1\}$. Based on (3), the arcs in \mathcal{E}_{010} can be partitioned as

$$\begin{aligned} \mathcal{E}_{010,0} &= \{ (\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}_{010}, \mathbf{y} |_{\mathcal{C}_{010}} = 0 \} = \{ (010, 000), \\ &\quad (011, 001), (012, 002), (110, 100), \\ &\quad (111, 101), (112, 102) \}, \\ \mathcal{E}_{010,1} &= \{ (\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}_{010}, \mathbf{y} |_{\mathcal{C}_{010}} = 1 \} = \{ (000, 010), \\ &\quad (001, 011), (002, 012), (100, 110), \\ &\quad (101, 111), (102, 112) \}. \end{aligned}$$

Similarly, the remaining arcs can be partitioned in the same manner. They are listed as follows.

$$\begin{split} \mathcal{E}_{001,0} &= \{(001,000), (101,100), (011,010), (111,110)\}, \\ \mathcal{E}_{001,1} &= \{(002,001), (102,101), (012,011), (112,111)\}, \\ \mathcal{E}_{001,2} &= \{(000,002), (100,102), (010,012), (110,112)\}, \\ \mathcal{E}_{002,0} &= \{(002,000), (102,100), (012,010), (112,110)\}, \\ \mathcal{E}_{002,1} &= \{(000,001), (100,101), (010,011), (110,111)\}, \\ \mathcal{E}_{002,2} &= \{(001,002), (101,102), (011,012), (111,112)\}, \\ \mathcal{E}_{100,0} &= \{(100,000), (110,010), (101,001), (111,011), \\ \end{split}$$

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Fig. 3. 4-regular digraph.

$$\mathcal{E}_{100,1} = \{(000, 100), (010, 110), (001, 101), (011, 111), (002, 102), (012, 112)\}.$$

If each partitioned subset is assigned with a distinct color, there will be 10 injective arc colors in the regular digraph, which is shown in Fig. 4. Based on *Theorem 1*, it can yield a (12, 12, 8, 10) PDA \mathbf{P}_0 that is defined at the bottom of the next page. It can be seen that \mathbf{P}_0 can realize a coded caching scheme with K = 12 users, the memory ratio $\frac{M}{N} = \frac{2}{3}$, the subpacketization level F = 12 and the transmission rate $R = \frac{5}{6}$. Note that for the MN scheme that yields the same number of users and memory ratio, its subpacketization level and transmission rate are 495 and $\frac{4}{9}$, respectively. Therefore, the coded caching scheme realized by \mathbf{P}_0 is advantageous in the subpacketization level, but sacrifice the transmission rate.

B. New PDA Constructions via Injective Arc Coloring of Regular Digraphs

The injective arc coloring of regular digraphs is employed for PDA design. Before presenting the new PDAs, we first introduce the following lemma, which is crucial to characterize the property of our proposed PDAs.

Lemma 2: For the digraph \hat{D} , $\chi_i(\hat{D}) \leq \sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} p_{i_0} p_{i_1} \cdots p_{i_{w-1}}$, where integers i_0, i_1, \dots, i_{w-1} are determined by $\mathcal{C}_{\mathbf{e}} = \{(j_{\alpha_0}^{(i_0)}, p_{i_0}), (j_{\alpha_1}^{(i_1)}, p_{i_1}), \dots, (j_{\alpha_{w-1}}^{(i_{w-1})}, p_{i_{w-1}})\}$.

Proof: Given any $\mathbf{e} \in \{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}$, based on (3), it can be seen that the arc set $\mathcal{E}_{\mathbf{e}} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} - \mathbf{y} = \mathbf{e}, (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}$ is partitioned into $p_{i_0}p_{i_1}\cdots p_{i_{w-1}}$ subsets since $|\mathcal{T}_{\mathbf{e}}| = p_{i_0}p_{i_1}\cdots p_{i_{w-1}}$. Hence, the total number of partitioned subsets is $\sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} |\mathcal{T}_{\mathbf{e}}| = \sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} p_{i_0}p_{i_1}\cdots p_{i_{w-1}}$. Based on *Proposition 1*, the number of colors that allow an injective arc coloring of \hat{D} is less than or equal to $\sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} p_{i_0}p_{i_1}\cdots p_{i_{w-1}}$. Integrating *Theorem 1* and *Lemma 2*, a new coded caching

scheme that is characterized by the following theorem can be obtained.

 caching scheme with a memory ratio of

$$\frac{M}{N} = 1 - \frac{\sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = w} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1)}{p_{\alpha}^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}}$$

and a transmission rate of

$$R = \frac{\sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} p_{i_0} p_{i_1} \cdots p_{i_{w-1}}}{p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}},$$

where

$$\mathcal{K} = \{\underbrace{p_0^{(0)}, p_0^{(1)}, \dots, p_0^{(n_0-1)}}_{n_0}, \underbrace{p_1^{(0)}, p_1^{(1)}, \dots, p_1^{(n_1-1)}}_{n_1}, \dots, \underbrace{p_{m-1}^{(n_m-1-1)}}_{n_{m-1}}, \dots, \underbrace{p_{m-1}^{(n_m-1-1)}}_{n_{m-1}}\}, \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_{m-1}}\}, \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}\}, \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}\}, \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}\}, \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}\}, \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}], \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}], \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}], \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}], \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}], \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}], \dots, \underbrace{p_{m-1}^{(n_m-1)}}_{n_m}]$$

and integers $i_0, i_1, \ldots, i_{w-1}$ are determined by $C_{\mathbf{e}} = \{(j_{\alpha_0}^{(i_0)}, p_{i_0}), (j_{\alpha_1}^{(i_1)}, p_{i_1}), \ldots, (j_{\alpha_{w-1}}^{(i_{w-1})}, p_{i_{w-1}})\}.$

Proof: Let \hat{D} denote a regular digraph defined in *Definition 6.* It can be seen that the number of vertices of \hat{D} is $|\mathcal{V}| = p_0^{n_0} p_1^{n_1} \dots p_{m-1}^{n_{m-1}}$. Given any vertex $\mathbf{y} \in \mathcal{V}$, it can be seen that the number of vertices $\mathbf{x} \in \mathcal{V}$ such that $d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = w$ is $\sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = w} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1)$. Therefore, \hat{D} is a regular digraph with both the indegree and outdegree being $\sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = w} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1)$. Together with the results of *Theorem 1* and *Lemma 2*, the conclusion can be reached.

In particular, let \overline{D} denote a regular digraph defined in *Definition* 6 with parameter m = 1. An upper bound on the injective chromatic index of \overline{D} can be obtained as follows, which can be viewed as a special case of *Lemma* 2.

Corollary 1: For the digraph \overline{D} , $\chi_i(\overline{D}) \leq \binom{n_0}{w} p_0^w (p_0 - 1)^w$. Proof: Let \overline{D} denote a digraph with vertex set $\mathcal{V} = \{\mathbf{x} = (x_0, x_1, \dots, x_{n_0-1}) \mid \mathbf{x} \in \mathbb{Z}_{p_0}^{n_0}\}$ and arc set $\mathcal{E} = \{(\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{x}) \mid d_H(\mathbf{x}, \mathbf{y}) = w, \mathbf{x}, \mathbf{y} \in \mathcal{V}\}$. Note that the cardinality of set $\{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}$ is $\binom{n_0}{w} (p_0 - 1)^w$. Based on (3), it can be seen that each $\mathcal{E}_{\mathbf{e}} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} - \mathbf{y} = \mathbf{e}, (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}$ is partitioned into p_0^w subsets since $|\mathcal{T}_{\mathbf{e}}| = p_0^w$. This implies that the total number of partitioned subsets is $\binom{n_0}{w} p_0^w (p_0 - 1)^w$, i.e., the number of colors that allow an injective arc coloring of \overline{D} is less than or equal to $\binom{n_0}{w} p_0^w (p_0 - 1)^w$.

Based on *Theorem 1* and *Corollary 1*, the following result can be obtained, which can be seen as a special case of *Theorem 2*.

Corollary 2: Given any $n_0, w, p_0 \in \mathbb{N}^+$ with $p_0 \geq 2$ and w < n, there always exists a $(p_0^{n_0}, p_0^{n_0}, p_0^{n_0}, p_0^{n_0} - \binom{n_0}{w}(p_0 - 1)^w, \binom{n_0}{w}p_0^w(p_0 - 1)^w)$ PDA which yields a $p_0^{n_0}$ -division $(p_0^{n_0}, M, N)$ coded caching scheme with a memory ratio of $\frac{M}{N} = 1 - \frac{\binom{n_0}{w}(p_0 - 1)^w}{p_0^{n_0}}$ and a transmission rate of $R = \frac{\binom{n_0}{w}(p_0 - 1)^w}{p_0^{n_0 - w}}$.

Proof: Let \overline{D} denote a regular digraph defined above with vertex set \mathcal{V} and arc set \mathcal{E} . Given any vertex $\mathbf{y} \in \mathcal{V}$, the number of vertices $\mathbf{x} \in \mathcal{V}$ such that $d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = w$ is $\binom{n_0}{w}(p_0 - 1)^w$. This implies that both the outdegree and indegree of each vertex are $\binom{n_0}{w}(p_0 - 1)^w$. Based on the results of *Theorem 1* and *Corollary 1*, the conclusion can be reached.

In fact, given the digraph \overline{D} with parameters n_0, w and p_0 such that $p_0 = 2$ and $n_0 \le 2w - 1$, the upper bound



Fig. 4. 4-regular digraph with the number of injective arc coloring of 10.

on injective chromatic index described in *Corollary 1* can be further improved by vertex coloring. A vertex coloring of a graph G is an assignment of colors to the vertices of G with one color for each vertex, so that the adjacent vertices are colored differently. The smallest number of colors needed in a proper vertex coloring of graph G is called the chromatic index of G, denoted by $\chi(G)$. It is well known that $\chi(G) \leq$ $1 + \Delta(G)$ [32], where $\Delta(G)$ is the maximal degree of G. In order to improve the upper bound on injective chromatic index of \overline{D} , we need the following lemma.

Lemma 3: Given the digraph \overline{D} with parameters n_0, w and p_0 such that $p_0 = 2$ and $n_0 \leq 2w - 1$, let the arcs of \overline{D} be partitioned as

$$\mathcal{E} = \bigcup_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} \mathcal{E}_{\mathbf{e}} = \bigcup_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} \bigcup_{i=0}^{g_{\mathbf{e}-1}} \mathcal{G}_{\mathbf{e}, \mathcal{D}_i}, \quad (4)$$

where $\mathcal{G}_{\mathbf{e},\mathcal{D}_i} = \{\mathcal{E}_{\mathbf{e},\mathbf{t}_i} \mid \mathbf{t}_i \in \mathcal{D}_i\}, \mathcal{D}_i \text{ is a subset of } \mathbb{Z}_2^w \text{ such that } d_{\mathrm{H}}(\mathbf{t}_i,\mathbf{t}_j) \geq n_0 - w + 1 \text{ for any } \mathbf{t}_i,\mathbf{t}_j \in \mathcal{D}_i, \text{ and } \mathbb{Z}_2^w = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{g_{\mathbf{e}}-1}.$ Consequently, assigning a distinct color for each subset $\mathcal{G}_{\mathbf{e},\mathcal{D}_i}$ forms an injective arc coloring.

Proof: It is sufficient to prove that any two arcs at a distance of exactly one in a directed path or in a common directed triangle receive distinct colors. Without loss of generality, for any two arcs $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \mathcal{G}_{\mathbf{e}, \mathcal{D}_i}$, if $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \mathcal{E}_{\mathbf{e}, \mathbf{t}_i}$, based on *Proposition 1*, they can be assigned with the same color. Now let us consider the case $(\mathbf{x}_1, \mathbf{y}_1) \in \mathcal{E}_{\mathbf{e}, \mathbf{t}_i}, (\mathbf{x}_2, \mathbf{y}_2) \in \mathcal{E}_{\mathbf{e}, \mathbf{t}_i}$ and $\mathbf{t}_i \neq \mathbf{t}_j$. One needs

to prove that arcs $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ can also be assigned with the same color. Based on (4), we obtain $\mathbf{t}_i = \mathbf{y}_1|_{\mathcal{C}_e}$ and $\mathbf{t}_j = \mathbf{y}_2|_{\mathcal{C}_e}$. Note that $d_H(\mathbf{t}_i, \mathbf{t}_j) \ge n_0 - w + 1$. There must exist a set $\mathcal{A} \subseteq \mathcal{C}_e$ with $|\mathcal{A}| \ge n_0 - w + 1$ such that $\mathbf{y}_1|_s \ne \mathbf{y}_2|_s$ for any $s \in \mathcal{A}$. Furthermore, it can be seen that $\mathbf{x}_1|_s = \mathbf{y}_2|_s$ and $\mathbf{x}_2|_s = \mathbf{y}_1|_s$ for $s \in \mathcal{A}$, since $\mathbf{x}_1 - \mathbf{y}_1 = \mathbf{x}_2 - \mathbf{y}_2 = \mathbf{e}$ and $p_0 = 2$. This implies that $d_H(\mathbf{x}_1, \mathbf{y}_2) = d_H(\mathbf{x}_1|_{\mathcal{C}_e}, \mathbf{y}_2|_{\mathcal{C}_e}) + d_H(\mathbf{x}_1|_{[0:n_0-1]\setminus\mathcal{C}_e}, \mathbf{y}_2|_{[0:n_0-1]\setminus\mathcal{C}_e}) \le w - (n_0 - w + 1) + n_0 - w = w - 1$. Therefore, $(\mathbf{x}_1, \mathbf{y}_2), (\mathbf{y}_2, \mathbf{x}_1) \notin \mathcal{E}$. Following a similar proof manner, it can also be concluded that $(\mathbf{x}_2, \mathbf{y}_1), (\mathbf{y}_1, \mathbf{x}_2) \notin \mathcal{E}$. Therefore, assigning a distinct color for each subset $\mathcal{G}_{e,\mathcal{D}_i}$ forms an injective arc coloring.

Lemma 4: Given the digraph D with parameters n_0, w and p_0 such that $p_0 = 2$ and $n_0 \le 2w - 1$, we have

$$\chi_i(\overline{D}) \le \begin{cases} \binom{n_0}{w} 2^{w-1}, \text{ if } n_0 = 2w - 1; \\ \binom{n_0}{w} (1 + \sum_{i=1}^{n_0 - w} \binom{w}{i}), \text{ if } n_0 < 2w - 1 \end{cases}$$

Proof: Let \overline{D} denote a digraph with vertex set $\mathcal{V} = \{\mathbf{x} = (x_0, x_1, \dots, x_{n_0-1}) \mid \mathbf{x} \in \mathbb{Z}_2^{n_0}\}$ and arc set $\mathcal{E} = \{(\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{x}) \mid d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = w, \mathbf{x}, \mathbf{y} \in \mathcal{V}\}$. Note that the arc partition of \overline{D} is $\mathcal{E} = \bigcup_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} \bigcup_{\mathbf{t} \in \mathbb{Z}_2^w} \mathcal{E}_{\mathbf{e}, \mathbf{t}}$, where $\mathcal{E}_{\mathbf{e}, \mathbf{t}} = \{(\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}, \mathbf{x} - \mathbf{y} = \mathbf{e}, \mathbf{y}|_{\mathcal{C}_{\mathbf{e}}} = \mathbf{t}\}$. If $n_0 < 2w - 1$, we can merge two subsets $\mathcal{E}_{\mathbf{e}, \mathbf{t}_i}$ and $\mathcal{E}_{\mathbf{e}, \mathbf{t}_j}$ for $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{Z}_2^w$ under the condition of $d_{\mathrm{H}}(\mathbf{t}_i, \mathbf{t}_j) \ge n_0 - w + 1$.

Now let us determine the number of colors that allow an injective arc coloring to \overline{D} . If $n_0 = 2w - 1$, we have $w = n_0 - w + 1 \le d_H(\mathbf{t}_i, \mathbf{t}_j) \le w$. This implies that for any $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{Z}_2^w$, two subsets $\mathcal{E}_{\mathbf{e},\mathbf{t}_i}$ and $\mathcal{E}_{\mathbf{e},\mathbf{t}_j}$ can be merged if and only if $d_H(\mathbf{t}_i, \mathbf{t}_j) = w$. That says for any $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{Z}_2^w$, two subsets $\mathcal{E}_{\mathbf{e},\mathbf{t}_i}$ and $\mathcal{E}_{\mathbf{e},\mathbf{t}_j}$ can be merged if and only if $\mathbf{t}_i + \mathbf{t}_j = \mathbf{1}$. Therefore, each $\mathcal{E}_{\mathbf{e}}$ is partitioned into 2^{w-1} subsets, where $\mathcal{E}_{\mathbf{e}} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} - \mathbf{y} = \mathbf{e}, (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}$. Note that the cardinality of set $\{\mathbf{x} - \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}$ is $\binom{n_0}{w}$. Hence, the total number of partitioned subsets is $\binom{n_0}{w} 2^{w-1}$. Based on *Lemma 3*, it can be seen that the number of colors that allow an injective arc coloring of \overline{D} is less than or equal to $\binom{n_0}{w} 2^{w-1}$.

If $n_0 < 2w - 1$, we can determine the injective arc coloring by using the vertex coloring of graph. Define a graph G with

		000	010	100	110	001	011	101	111	002	012	102	112
	000 /	′ *	1	9	*	6	*	*	*	4	*	*	*)
	010	0	*	*	9	*	6	*	*	*	4	*	*
	100	8	*	*	1	*	*	6	*	*	*	4	*
	110	*	8	0	*	*	*	*	6	*	*	*	4
	001	2	*	*	*	*	1	9	*	7	*	*	*
р	011	*	2	*	*	0	*	*	9	*	7	*	*
$\mathbf{P}_0 =$	101	*	*	2	*	8	*	*	1	*	*	7	* .
	111	*	*	*	2	*	8	0	*	*	*	*	7
	002	5	*	*	*	3	*	*	*	*	1	9	*
	012	*	5	*	*	*	3	*	*	0	*	*	9
	102	*	*	5	*	*	*	3	*	8	*	*	1
	112	*	*	*	5	*	*	*	3	*	8	0	* /

vertex set $\mathcal{V} = [0:1]^w$ such that there exists an edge between two vertices \mathbf{t}_i and \mathbf{t}_j in \mathcal{V} if and only if $d_{\mathrm{H}}(\mathbf{t}_i, \mathbf{t}_j) \leq n_0 - w$. Given any vertex $\mathbf{t}_i \in \mathcal{V}$, the number of vertices of G that are adjacent to \mathbf{t}_i is $\sum_{i=1}^{n_0-w} {w \choose i}$, i.e., G is a $\sum_{i=1}^{n_0-w} {w \choose i}$ -regular graph. As a result, one color can be assigned to each vertex of G such that all adjacent vertices receive distinct colors. This indicates that $d_{\mathrm{H}}(\mathbf{t}_i, \mathbf{t}_j) \geq n_0 - w + 1$ for any two vertices \mathbf{t}_i and \mathbf{t}_j with the same color, i.e., two subsets $\mathcal{E}_{\mathbf{e},\mathbf{t}_i}$ and $\mathcal{E}_{\mathbf{e},\mathbf{t}_j}$ can be merged. Based on the upper bound of chromatic number of G, it can be seen that each $\mathcal{E}_{\mathbf{e}}$ is partitioned into $\chi(G)$ subsets, where $\chi(G) \leq 1 + \sum_{i=1}^{n_0-w} {w \choose i}$. This implies that the total number of partitioned subsets is less than or equal to ${n \choose w}(1 + \sum_{i=1}^{n_0-w} {w \choose i})$. Based on Lemma 3, it can be seen that the number of colors that allow an injective arc coloring of \overline{D} is less than or equal to ${n \choose w}(1 + \sum_{i=1}^{n_0-w} {w \choose i})$.

Combined with *Theorem 1* and *Lemma 4*, the following coded caching scheme that achieves a smaller transmission rate can be obtained.

Theorem 3: Given any $n_0, w, p_0 \in \mathbb{N}^+$ with $p_0 = 2$ and $w < n_0$, there always exists a 2^{n_0} -division $(2^{n_0}, M, N)$ coded caching scheme with a memory ratio of $\frac{M}{N} = 1 - \frac{\binom{n_0}{w}}{2^{n_0}}$ and a transmission rate of

$$R = \begin{cases} \frac{\binom{n_0}{2^{n_0 - w + 1}}, \text{if } n_0 = 2w - 1;\\ \frac{\binom{n_0}{w}(1 + \sum_{i=1}^{n_0 - w} \binom{w}{i})}{2^{n_0}}, \text{if } n_0 < 2w - 1 \end{cases}$$

In fact, if each colored edge of a regular graph is replaced by two reverse arcs with the same color, the strong edge coloring of regular graphs can be viewed as a special injective arc coloring of regular digraphs, as illustrated by Fig. 5. Therefore, based on *Theorem 1*, some new coded caching schemes with a linear subpacketization level can be obtained from some existing strong edge coloring of regular graphs. A unitary Cayley graph is a graph with vertex set \mathbb{Z}_n and edge set $\mathcal{E} = \{\{i, j\} \mid \gcd(i-j, n) = 1, i, j \in \mathbb{Z}_n\}$, where $\gcd(i-j, n)$ denotes the greatest common divisor between integers i - jand n. It can be observed that a unitary Cayley graph is a regular graph with degree $\psi(n)$, where $\psi(n)$ denotes the Euler function, i.e., the number of integers that are less than n and relatively prime to n. The strong chromatic index of Cayley graphs is characterized as follows.

Lemma 5 [34]: Given any positive integer $n = p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}$ with prime factor $p_i \ge 2$ for $i \in [0:m-1]$, the strong chromatic index of unitary Cayley graphs is $\frac{n\psi(n)}{2^m}$.

Based on *Theorem 1* and *Lemma 5*, we have the following result.

Theorem 4: Given any positive integer $n = p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}$ with prime factor $p_i \ge 2$ for $i \in [1:m-1]$, there always exists an $(n, n, n - \psi(n), \frac{n\psi(n)}{2^m})$ PDA which yields an *n*-division (n, M, N) coded caching scheme with a memory ratio of $\frac{M}{N} = 1 - \frac{\psi(n)}{n}$ and a transmission rate of $R = \frac{\psi(n)}{2^m}$.

It can be seen that the coded caching schemes characterized in *Theorems 2* and 3 and *Corollary 2* require a high memory ratio, even though they all exhibit a linear subpacketization level. In the following subsection, we will show that the memory ratios and subpacketization levels of the schemes in *Theorem 2* and *Corollary 2* can be further reduced by using the MDS code in the placement phase.

C. New PDA Schemes With Coded Placement

In a PDA, a "*" is called useless, if it is not contained in any subarray shown in C3-(b) of *Definition 1*. This indicates these useless "*"s cannot generate multicasting opportunities in the delivery phase, i.e., they have no contributions in reducing the transmission rate of a coded caching scheme realized by the PDA and they result in both a high memory ratio and a high subpacketization level. Therefore, if each column of a (K, F, Z, S) PDA has Z' useless "*"s, we can obtain a new coded caching scheme with a smaller memory ratio and subpacketization level by deleting these useless "*"s and using an $(F, F - Z')_q$ MDS code that is defined in a finite field of size q. For the detailed implementation method, interested readers can refer to [21].

Lemma 6 [21]: For any (K, F, Z, S) PDA **P**, if there exist Z' useless "*"s in each column, we can obtain an (F - Z')-division (K, M, N) coded caching scheme with a memory ratio of $\frac{M}{N} = \frac{Z-Z'}{F-Z'}$ and a transmission rate of $R = \frac{S}{F-Z'}$. The coding gain at each time slot is the same as the original scheme realized by **P**.

Proof: If there exist Z' useless "*"s in each column of the (K, F, Z, S) PDA **P**, a new array $\mathbf{P}' = (\mathbf{P}'[j, k])$, where $j \in [0: F-1]$ and $k \in [0: K-1]$, can be obtained by deleting the Z' useless "*"s in each column of **P**. As a result, each column of \mathbf{P}' has Z' blanks, F - Z integers and Z - Z' "*"s. Based on P', the placement strategy in Algorithm 1 can be modified as follows. Each file is divided into F - Z' equal packets. They are then encoded by using an $(F, F - Z')_q$ MDS code. The encoded packets are denoted as $W'_{n,0}, \ldots, W'_{n,F-1}$ for each file W_n , where $n \in [0: N-1]$. Based on the caching strategy of Algorithm 1, each user kcaches $Z_k = \{W'_{n,j} \mid \mathbf{P}'[j,k] = *, j \in [0 : F-1], n \in [0 : N-1]\}$. Therefore, the memory ratio of each user is $\frac{M}{N} = \frac{Z-Z'}{F-Z'}$. In the delivery phase, we also use the delivery strategy of Algorithm 1. For any request vector **d**, based on Lines 6-7 of Algorithm 1 and the modified placement strategy, each user can obtain F - Z' required coded packets. The $(F, F - Z')_a$ MDS code guarantees that each user can reconstruct its requested file after receiving F - Z' coded symbols. Therefore, the transmission rate is $R = \frac{S}{F - Z'}$.

Note that the operation field size q of Lemma 6 is $\mathcal{O}(F)$, implying that the size of each packet of files should approximate to $\log_2 F$ bits. Therefore, the size of files in the server must be greater than $(F - Z') \log_2 F$ so that the transmission rate of $\frac{S}{F-Z'}$ can be achieved. Given a (K, F, Z, S) PDA, $\frac{Z}{F} > \frac{Z-Z'}{F-Z'}$ and F > F - Z' always hold for any $Z, F, Z' \in \mathbb{N}^+$. Therefore, the scheme in Lemma 6 has a smaller memory ratio and subpacketization level than that of Lemma 1. Furthermore, the scheme in Theorem 2 can be improved as follows.

Theorem 5: Given any $w, n_i, m \in \mathbb{N}^+$ and distinct positive integers $p_0, p_1, \ldots, p_{m-1}$ with $p_i \geq 2$ and $w < n_0 + n_1 + \cdots + n_{m-1}$ for $i \in [0 : m-1]$, there always exists a $[p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}} - (1 + \sum_{i=1}^{w-1} \sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = i} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} \sum_{i=1}^{m_{\alpha}} \sum_{j \in \mathcal{X}} \sum_{i=1}^{m_{\alpha}} \sum_{j \in \mathcal{X}} \sum_{i=1}^{m_{\alpha}} \sum_{j \in \mathcal{X}} \sum_{j \in \mathcal{X}} \sum_{i=1}^{m_{\alpha}} \sum_{j \in \mathcal{X}} \sum_$



Fig. 5. (a) 3-regular graph with the number of strong edge coloring of 6; (b) 3-regular digraph with the number of injective arc coloring of 6.

 $(p_{\alpha} - 1))$]-division $(p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}, M, N)$ coded caching scheme with a memory ratio of, as shown in the first equation at the bottom of the next page, and a transmission rate of, as shown in the second equation at the bottom of the next page, where

$$\mathcal{X} = \{\underbrace{p_0^{(0)}, p_0^{(1)}, \dots, p_0^{(n_0-1)}}_{n_0}, \underbrace{p_1^{(0)}, p_1^{(1)}, \dots, p_1^{(n_1-1)}}_{n_1}, \dots, \underbrace{p_{m-1}^{(0)}, p_{m-1}^{(1)}, \dots, p_{m-1}^{(n_{m-1}-1)}}_{n_{m-1}}\},$$

and integers i_0, i_1, \dots, i_{w-1} are determined by $C_{\mathbf{e}} = \{(j_{\alpha_0}^{(i_0)}, p_{i_0}), (j_{\alpha_1}^{(i_1)}, p_{i_1}), \dots, (j_{\alpha_{w-1}}^{(i_{w-1})}, p_{i_{w-1}})\}.$

Proof: Let **P** denote an array generated from the injective arc-colored regular digraph \hat{D} . Based on the proof of **Proposition 1**, it can be seen that a "*" in entry $\mathbf{P}[\mathbf{x}, \mathbf{y}]$ is useless if and only if $d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) < w$, where $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. Therefore, the number of useless "*"s in each column of **P** is

$$Z' = \begin{cases} 1, & \text{if } w = 1; \\ 1 + \sum_{i=1}^{w-1} \sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = i} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1), & \text{if } w > 1. \end{cases}$$

Combined with the results of *Lemma 6* and *Theorem 2*, the conclusion can be reached.

Continued from *Example 4*, the following example further illustrates the realization of the scheme in *Theorem 5*.

Example 5: Let \mathbf{P}_0 denote the PDA generated by the regular digraph of Fig. 4. It can be seen that a "*" of \mathbf{P}_0 is useless if and only if $d_{\rm H}(\mathbf{x}, \mathbf{y}) < 1$, where \mathbf{x} and \mathbf{y} are the vertices of the regular digraph. Hence, each column of \mathbf{P}_0 has one useless "*"s, i.e., the "*"s in the main diagonal positions of \mathbf{P}_0 are useless. Let \mathbf{P}'_0 denote an array obtained by deleting these useless "*"s. Based on *Lemma* 6, \mathbf{P}'_0 can realize a coded caching scheme with K = 12 users and subpacketization level of F = 11. Meanwhile, it yields a memory ratio of $\frac{M}{N} = \frac{7}{11}$ and the transmission rate of $R = \frac{10}{11}$, which are in line with *Theorem 5*.

Based on *Corollary 2* and *Lemma 6*, the following corollary can be obtained, which can be seen as a special case of *Theorem 5*. Since its proof is similar to that of *Theorem 5*, it is omitted.

Corollary 3: Given any $n_0, w, p_0 \in \mathbb{N}^+$ with $p_0 \ge 2$ and $w < n_0$, there always exists a $[p_0^{n_0} - \sum_{i=0}^{w-1} {n_0 \choose i} (p_0 - 1)^i]$ -division $(p_0^{n_0}, M, N)$ coded caching scheme with a memory

ratio of $\frac{M}{N} = 1 - \frac{\binom{n_0}{p_0^{n_0} - \sum_{i=0}^{w-1} \binom{n_0}{i} (p_0 - 1)^w}}{p_0^{n_0} - \sum_{i=0}^{w-1} \binom{n_0}{i} (p_0 - 1)^i}$ and a transmission rate of $R = \frac{\binom{n_0}{w} p_0^w (p_0 - 1)^w}{p_0^{n_0} - \sum_{i=0}^{w-1} \binom{n_0}{i} (p_0 - 1)^i}.$

To further show that the proposed coded caching schemes of *Theorems 4* and 5 can yield a small memory ratio, Fig. 6 plots how their memory ratios range as the number of users increase. In most cases, a memory ratio of less than 0.5 can be achieved.

Finally, it should be pointed out that if we consider a regular digraph with vertex set $\mathcal{V} = \mathbb{Z}_2^n$ (or \mathbb{Z}_3^n) and arc set $\mathcal{E} = \{(\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{x}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{V}, d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = w\}$, the PDA constructions proposed in [24] can also be viewed as an application of *Theorem 1*. The authors designed an appropriate partition for the entries of an array to satisfy the PDA constraints. Its partition rule can be seen as the equivalent class of edges defined in [35]. Unlike previous existing constructions, our proposed construction of PDAs depends on the injective arc-colored regular digraphs, and the corresponding arc partition rule is different with the one of [35]. Furthermore, our proposed coded caching scheme in *Theorem 5* extends the scheme of [24] to the case with a flexible number of users.

V. PERFORMANCE ANALYSES OF THE NEW SCHEMES

This section analyzes the proposed coded caching schemes in terms of the subpacketization level and transmission rate. They are compared with the existing schemes in Table I.

A. Comparison Between the Schemes in Theorem 5, Corollary 3 and [1], [16], [17], [24]

We first consider the comparison between the schemes in *Theorem 5* and [1], [17]. Note that the expressions of the memory ratio, transmission rate, and subpacketization level are complex for the scheme in *Theorem 5*. It is challenging to yield a conclusive analysis for its performance. Alternatively, we numerically compare it with the existing ones of [1] and [17] by taking the number of users K = 1536, 3072. Figs. 7 and 8 characterize their subpacketization level and transmission rate performances against the memory ratio. It can be seen that with some sacrifice in the transmission rate, our proposed scheme in *Theorem 5* significantly reduces the subpacketization level of the MN scheme. Meanwhile, for some memory ratios that are greater than 0.5, our proposed



Fig. 6. The minimum memory ratio $\frac{M}{N}$ vs. the number of users K of the schemes in *Theorems 4* and 5.

scheme in *Theorem 5* has advantages in both the subpacketization level and transmission rate when compared with the MN scheme with grouping in [17].

We then consider the comparison between the schemes in *Corollary 3* and [24]. With $p_0 = 2$, a coded caching scheme in *Corollary 3* can yield $K = 2^{n_0}$, $\frac{M}{N} = 1 - \frac{\binom{n_0}{w}}{\sum_{i=w}^{n_0} \binom{n_0}{i}}$, $F = \sum_{i=w}^{n_0} \binom{n_0}{i}$, $R = \frac{\binom{n_0}{w}2^w}{\sum_{i=w}^{n_0} \binom{n_0}{i}}$. By letting $w' = n_0 - w$ for the scheme of [24] (which is also shown in Table I), it can be observed that when $p_0 = 2$, it will be the same as the scheme in *Corollary 3*. Therefore, our scheme proposed in *Corollary 3* generalizes the scheme of [24] and accommodates a more flexible number of users.

We further compare our proposed scheme in *Corollary 3* with the scheme of [16] in Table II. The parameters of the scheme in *Corollary 3* are parameterized by (n_0, w, p_0) . It can be seen that the proposed scheme in *Corollary 3* has a smaller subpacketization level, a slightly smaller memory ratio and a lower transmission rate. Meanwhile, it can support more users.

B. Comparison Between the Schemes in Theorem 4 and [1], [11]

We compare our proposed scheme in *Theorem 4* with the schemes of [1] and [11] in Table III. Note that the schemes in

TABLE II COMPARISON BETWEEN THE SCHEME IN COROLLARY 3 AND THE SCHEME IN [16]

Schemes	Parameters	K	F	$\frac{M}{N}$	R
(n_0, w, p_0) in Corollary 3	(7, 6, 6)	279936	187500	0.42	27216.0
(m, a, b, λ) in [16]	(40, 36, 34, 30)	91390	3838380	0.49	46376.0
(n_0, w, p_0) in Corollary 3	(8, 7, 6)	1679616	1015625	0.38	172270.0
(m, a, b, λ) in [16]	(54, 50, 48, 44)	31625	2587165	0.38	194580.0
(n_0, w, p_0) in Corollary 3	(8, 6, 4)	65536	44469	0.54	1880.1
(m, a, b, λ) in [16]	(26, 22, 20, 16)	14950	230230	0.68	4845.0
(n_0, w, p_0) in Corollary 3	(6, 4, 5)	15625	14080	0.73	170.5
(m, a, b, λ) in [16]	(20, 16, 14, 10)	4845	38760	0.79	1001.0

TABLE III Comparison Between the Scheme in Theorem 4 and the Schemes in [1] and [11]

	n			M	D
Schemes	Parameters	K	F'	$\frac{M}{N}$	R
(n, m, k, q) in [11]	(2, 2, 4, 2)	105	105	0.54	8.0
(n) in Theorem 4	(105)	105	105	0.54	6.0
(k,t) in [1]	(105, 57)	105	2.1×10^{30}	0.54	0.8
(n, m, k, q) in [11]	(2, 3, 5, 2)	465	4340	0.59	19.2
(n) in Theorem 4	(465)	465	465	0.48	30.0
(k,t) in [1]	(465, 223)	465	inf.	0.48	1.1
(n, m, k, q) in [11]	(2, 4, 6, 2)	1953	546840	0.61	51.2
(n) in Theorem 4	(1953)	1953	1953	0.45	135.0
(k,t) in [1]	(1953, 879)	1953	inf.	0.45	1.2
(n, m, k, q) in [11]	(1, 5, 6, 2)	63	5249660	0.49	5.3
(n) in Theorem 4	(63)	63	63	0.43	9.0
(k,t) in [1]	(63, 27)	63	$ 4.9 \times 10^{17} $	0.43	1.3

Theorem 4 and [1], [11] are parameterized by (n), (k, t) and (n, m, k, q), respectively. Table III shows that in comparison with the scheme of [11], for the same number of users, subpacketization level and memory ratio, our proposed scheme in *Theorem* 4 has an advantage in the transmission rate. Moreover, with the same number of users and a slightly smaller memory ratio, our proposed scheme yields a smaller subpacketization level than that of the scheme in [1] and [11]. But they are realized at the cost of some transmission rate.

C. Comparison Between the Schemes in Theorem 3, [16] and [23]

We discuss the performance of our scheme in *Theorem 3* by comparing it with the ones of [16] and [23]. The parameters

$$\frac{M}{N} = \begin{cases}
1 - \frac{\sum_{i=0}^{m-1} n_i(p_i - 1)}{p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}} - 1}, & \text{if } w = 1; \\
1 - \frac{\sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = w} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1)}{p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}} - 1 - \sum_{i=1}^{w-1} \sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = i} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1)}, \\
& \text{if } w > 1,
\end{cases}$$

$$R = \begin{cases} \frac{\sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} p_{i_0}}{p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}} - 1}, & \text{if } w = 1; \\ \frac{\sum_{\mathbf{e} \in \{\mathbf{x} - \mathbf{y} | (\mathbf{x}, \mathbf{y}) \in \mathcal{E}\}} p_{i_0} p_{i_1} \cdots p_{i_{w-1}}}{p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}} - (1 + \sum_{i=1}^{w-1} \sum_{\mathcal{A} \subseteq \mathcal{X}, |\mathcal{A}| = i} \prod_{p_{\alpha}^{(\beta)} \in \mathcal{A}} (p_{\alpha} - 1))}, \\ & \text{if } w > 1, \end{cases}$$



Fig. 7. Subpacketization level and transmission rate comparison between the schemes of *Theorem 5* and [1], [17]. With K = 1536, for the scheme in *Theorem 5*, $p_0 = 2, p_1 = 3, n_0 = 9, n_1 = 1, m = 2$ and $w \in [2:9]$; for the scheme of [1], k = 1536 and $t \in [1:1535]$; for the grouping scheme in [17], k = 24, c = 64 and $t \in [1:23]$.



Fig. 8. Subpacketization level and transmission rate comparison between the schemes in *Theorem 5* and [1], [17]. With K = 3072, for the scheme in *Theorem 5*, $p_0 = 2$, $p_1 = 3$, $n_0 = 10$, $n_1 = 1$, m = 2 and $w \in [2:10]$; for the scheme in [1], k = 3072 and $t \in [1:3071]$; for the grouping scheme in [17], k = 24, c = 128 and $t \in [1:23]$.

TABLE IV Comparison Between the Scheme in Theorem 3 and the Schemes in [16] and [23]

Schemes	Parameters	K	F	$\frac{M}{N}$	R
(m, a, b, λ) in [16]	(16, 12, 10, 6)	1820	8008	0.88	210.0
(r,k,z) in [23]	(5, 256, 1)	4096	4096	0.81	95.7
$ (n_0, w)$ in Theorem 3	(13,7)	4096	4096	0.81	23.2
(m, a, b, λ) in [16]	(21, 16, 7, 4)	20349	116280	0.84	212.0
(r,k,z) in [23]	(6, 512, 1)	32768	32768	0.89	56.0
$ (n_0, w)$ in Theorem 3	(15, 8)	32768	32768	0.80	25.1
(m, a, b, λ) in [16]	(20, 14, 12, 7)	38760	125970	0.84	792.0
(r,k,z) in [23]	(6, 1024, 1)	65536	65536	0.89	111.9
$ (n_0, w)$ in Theorem 3	(16, 9)	65536	65536	0.83	87.6
(m, a, b, λ) in [16]	(26, 16, 12, 5)	5311700	9657700	0.95	5148.0
(r,k,z) in [23]	(7, 65536, 1)	8388608	8388608	0.94	4096.0
(n_0, w) in Theorem 3	(23, 15)	8388608	8388608	0.94	1338.0

of the schemes in *Theorem 3*, [16] and [23] are written as (m, a, b, λ) , (n_0, w) and (r, k, z), respectively. Table IV shows that in comparison with the schemes of [16], our proposed

scheme in *Theorem 3* yields a smaller subpacketization level, a slightly smaller memory ratio and a lower transmission rate. Meanwhile, it is capable to serve more users. When comparing with the scheme of [23], with the same number of users and subpacketization level, our proposed scheme in *Theorem 3* has transmission rate advantage.

VI. CONCLUSION

This paper has investigated the design of a PDA with the same number of rows and columns through the perspective of graph coloring, i.e., the injective arc coloring of a regular digraph. From this perspective, designing coded caching schemes with a linear subpacketization level can be converted into determining the number of colors that allow an injective arc coloring to the regular digraphs. Based on this comprehension, we have defined a new class of regular digraphs and derived the upper bounds for the digraphs' injective chromatic index. Consequently, three new coded caching schemes that can support a flexible number of users have been obtained with a linear subpacketization level and a relatively small transmission rate.

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